Ministry of Education Ontario
Minister of Education Hon. Thomas L. Wells
A Support Document to *The Formative Years*
1976

This document describes concepts that are prerequisite for computation and outlines some strategies for developing, maintaining, and extending the skills of calculation.

From Counting to Calculation

Introduction

The effectiveness of a school's mathematics program has long been judged on the basis of tests purporting to measure basic skills in arithmetic. This situation resulted partly from the fact that facility in computation was regarded as a valuable asset in the job market, and thus a legitimate objective for education, and partly from the fact that arithmetic skills appear relatively easy to measure. Although these reasons for emphasizing basic arithmetic skills may not be as valid today, arithmetic is still an important component of the mathematics program.

Numbers have more significance than ever before for virtually everyone, and so it is important for children to learn how they are formed, how to operate with them, when to use them, and how to interpret the results. The ability to do arithmetic competently depends on the development of certain mental structures in the child. How and when these structures are established depends on the variety of the child's experiences as well as on his particular stage of development. This document describes concepts that are prerequisite for computation and then outlines some strategies for maintaining and extending the skills of calculation.

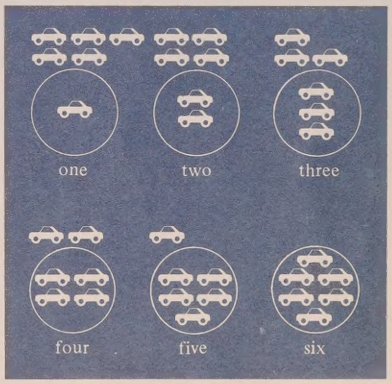
Counting

Counting is the process that answers the question "How many? " Many children come to school able to recite number names; teachers should not mistake this for the ability to count. In counting, words are used to record the number of objects in a collection, in the same way that notches on a stick or strokes on a clay tablet were used in ancient times.

The child must learn the unique name of each number and be able to recall these names in a specific order; then he must apply these names in the correct order to the objects he is counting. Although the order of the number names is important, the child should understand that the assignment of particular names to particular objects is not. This point is illustrated by the following example.

A boy was counting six toy cars. He picked up each in turn saying, 'One, two, three, four, five, six.' Accidentally, two of the cars fell to the floor. Seeing this, his mother asked, 'How many cars do you have now?' The boy replied, 'One, two, five, six,' picking up each of the remaining cars in turn. The mother then said, 'You only have four cars, one, two, three, four.' The boy, quite upset, responded, 'Three and four are on the floor.'

In this example, even though the boy said the cardinal words *one, two, three, four, five, six* , he has associated the ordinal idea of *first, second, third fourth, fifth* and *sixth* with the cars. He is correct in realizing that the third and fourth cars are missing, but has missed the collective aspect of counting. This example shows that the boy needs a number of experiences in which he can associate the counting numbers with sets of different sizes, as illustrated below.



Children often confuse the cardinal and ordinal aspects of numbers. It is important to provide experiences that enable them to discriminate between these two uses of numbers. Often the nature of the objects chosen contributes to the misunderstanding, as is the case in the following story.

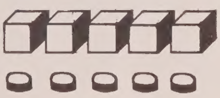
The principal was asked to confirm a teacher's assessment that an eight-year-old girl needed special help with her counting. After preliminary discussion, the principal asked the girl to count the fingers on his hand. She responded by touching each finger in turn, saying, 'One, two, three, four, five.' The principal then asked her to count his fingers in the other direction. The girl began 'One, two,' and then stopped. Puzzled, she asked, 'How can it be both two and four? '

The idea of associating the four fingers collectively with the number name *four* had not yet been distinguished from the ordinal idea of associating the number name *four* with the fourth finger in the sequence. Obviously, this child needed many more experiences counting real things that are not as fixed in position as the fingers on a hand. Counting objects in a set naturally invites ordinal associations. The number names are applied to successive objects in the set until all the objects in the set are exhausted. Then the number applied to the *last* object (ordinal idea) in the set corresponds to the cardinal number of the set.

To understand numbers fully, children need both the idea of the number of things in a collection and the idea of the order or sequence of things. These two concepts could be developed in the following ways:
(i) children could be asked to build a "staircase" out of rods of progressively increasing size;
(ii) children could be asked to match the individual items of two sequences, both of which consist of objects of increasing size (e.g., dolls and walking sticks);
(iii) children could be asked to explain the concept that *six* is *one more* than *five*.

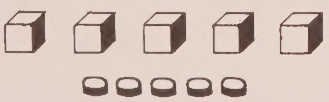
Conservation of Number
Once a child can count the objects in a set without difficulty, it is important to find out if he understands the concept of conservation of number – that is, does he believe that the number remains the same regardless of the spatial arrangement of the objects? There are many ways to test this using two sets of objects of differing characteristics, such as coins and blocks.

Test of Conservation
Place 5 blocks and 7 coins on the table. Ask the child to match the blocks and coins. Remove the surplus coins.



Question: Do you have the same number of pennies as you have blocks? – or are there more pennies? – or more blocks? How do you know this?

If the child says that the two sets have the same number, change the arrangement by spreading out the blocks.



Question: Now, do you have the same number of pennies as you have blocks? – or are there more pennies? – or more blocks? Why?

If the child's answer shows that he can conserve number at this level, it is suggested that he be tested again using 12 to 15 objects.

Notation
Some teachers expect children to recognize and write numerals early in their school experience. This may not always be a wise expectation as children often learn to use these symbols as labels without attaching real meaning to them.

Children need to pass through several levels of experience in learning to record numbers and use notation. First, they need many practical experiences to help them develop facility in counting and naming groups orally. From these situations they will develop their own system of notation for recording their ideas about numbers. This process begins with the experience of matching number names to objects such as pebbles or counters, is followed by the "tally" system, and culminates in making marks on paper to match the groups observed and regrouping these marks in twos, threes, fours, and fives. Finally, numerals and symbols for operations can be developed as the standard form through which people communicate ideas about numbers.

Children can usually "read" numerals before they can write them. This makes it all the more important for them to associate the written numerals with the verbal labels and with a variety of experiences in counting objects.

As soon as the child has a knowledge of the numbers from one to ten, he should be encouraged to use symbols to record this knowledge: $1 + 5 = 6$, $2 + 2 = 4$, $3 + 6 = 9$, $7 - 3 = 4$, and so on.

Practice in writing numerals is necessary for clear communication. This activity has the same relationship to understanding numbers as copying the alphabet has to reading.

Number Relationships
Counting real objects in a variety of situations enables children to discover many number relationships. For younger children these experiences should be limited to numbers up to 7. For some, the activity must be repeated over and over again before patterns emerge. Children must learn for themselves that it is not necessary to count on in ones to find that the sum of 4 and 2 is always 6. In fact, it is through the manipulation of concrete materials that it becomes obvious that $4 + 2$ and $2 + 4$ have the same sum. This discovery greatly reduces the number of addition facts to be memorized.

When the child has acquired an understanding of the numbers up to 7, he is ready to extend this study to 10 and beyond. It should be noted that these extensions do not require equal amounts of work or time to develop. Once the child has accepted the number system as the union of classification and ordering, the major difficulty with large numbers lies in the notation or, in this case, place value.

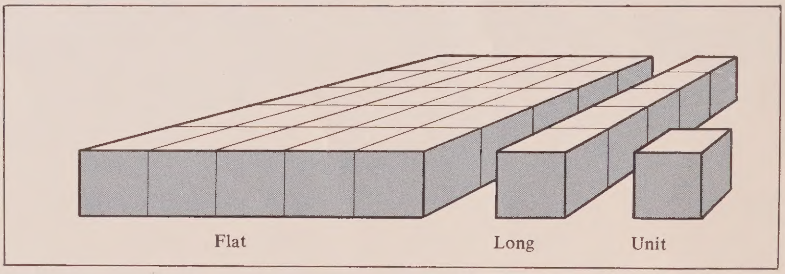
Place Value
Place value is one of the most important concepts related to counting and recording numbers. Effective teaching of this concept involves the use of manipulative materials (straws, discs, beads, or tongue depressors) to illustrate different groupings. Cubes, however, constitute the best model for demonstrating place value; their faces are congruent squares that can be fitted exactly on all sides.

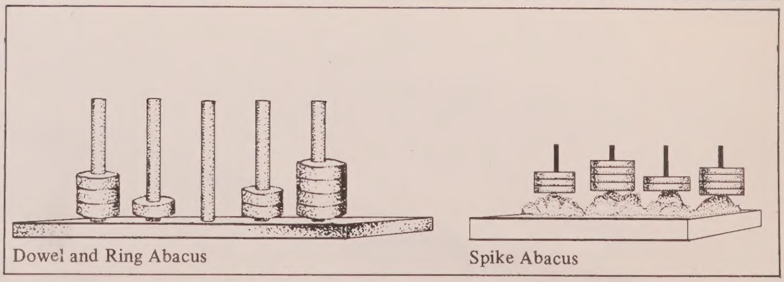
Dr. Zoltan Dienes has worked extensively with young children using his multibase arithmetic blocks to introduce numeration bases. Evidence collected over many years indicates that children of 7 or 8 years can use these blocks with ease to organize numbers in several different bases. This experience helps them to better understand the organizational pattern of our decimal system of notation.

The abacus is another useful device for the representation of numbers. It can easily be improvised from scrap materials such as bits of wire or wooden dowels and washers or rings.

Children's ability to read and write numbers with both understanding and confidence depends on the variety of concrete forms in which they can represent the number system.

Dienes' Multibase Blocks





Structural Materials

As children begin to organize their knowledge of numbers, it is useful to introduce them to a variety of structural materials. The size, and in some cases the colour, of the materials is a useful aid in helping the child arrange the apparatus into meaningful patterns. For example, he can easily build for himself all the number relationships whose sum is 8 and record these on a chart such as the one given below.

8								
7	8							
6		8						
5			8					
4				8				
3					8			
2						8		
1							8	
+	1	2	3	4	5	6	7	8

The chart can be extended to 20 and used as a ready reference, if needed. This arrangement emphasizes many of the patterns that will help the child remember individual facts.

10	11	12	13	14	15	16	17	18	19	20
9	10	11	12	13	14	15	16	17	18	19
8	9	10	11	12	13	14	15	16	17	18
7	8	9	10	11	12	13	14	15	16	17
6	7	8	9	10	11	12	13	14	15	16
5	6	7	8	9	10	11	12	13	14	15
4	5	6	7	8	9	10	11	12	13	14
3	4	5	6	7	8	9	10	11	12	13
2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11
+	1	2	3	4	5	6	7	8	9	10

Basic Number Facts
Test

A good way to find out if a child knows the basic facts is to ask him orally. Brief, frequent quizzes enable the teacher to determine:

(i) whether each child is confident of his knowledge or whether he relies on 'counting on';

(ii) whether he understands that he can add numbers in any order (3+7 or 7+3) and whether he uses this to his advantage;

(iii) whether he recognizes and uses patterns that relate operations with 9 or 11 to operations with 10;

(iv) whether he uses estimation as a preliminary guide (e.g., 18 is 'near enough' to 20; 998 is 'near enough' to 1000) before working out the exact answer.

The extension of the child's knowledge of number relationships from 20 to 100 is a lengthy process. Again, it is useful to work through number patterns such as $8 + 7 = 15$, $18 + 7 = 25$, $28 + 7 = 35$, and so on. These, along with those for $95 - 8 = 87$, $85 - 8 = 77$, $75 - 8 = 67$, and so on, need to be discovered by each child. Once these patterns are understood, they need to be reinforced through regular oral practice. There are also many interesting games that can be used either individually or in small groups to develop the child's competence and confidence. As with smaller numbers, a good way to find out whether children know facts about larger numbers is to ask individuals orally.

Number Line

A number line is a useful device to use at this stage. It can be made from graph paper or cash register tape. Fastened horizontally to the wall at a level within reach of the children, it should be at least 200 units long. The children can use coloured cardboard strips from 1 unit to 20 units in length to see number patterns such as 3 + 8, 13 + 8, 23 + 8, and so on.

Computation

Pages 70-71 of *Education in the Primary and Junior Divisions* outline criteria for determining the readiness of each child for practice in written computation. It should be expected that the level of proficiency attainable by different children will vary widely. Thus care must be taken to choose activities that favour a reasonable expectation of success even though some children may have to use aids such as tables, number lines, and 'ready reckoners'. The inefficiency of these aids can provide the motivation for the children to memorize the facts they need most often.

The problems that emerge in the course of investigations involving real materials can lead quite naturally to written calculations. At first, the children should be encouraged to devise their own methods of working out the answer. These will probably be primitive, but through discussion within the group and questions from the teacher, the methods can soon be refined. The following anecdote illustrates this process.

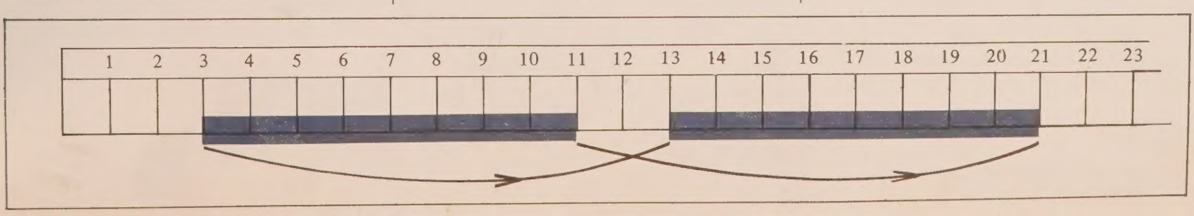
A teacher of ten- to eleven-year-old children was just beginning to establish activity centres for problem-solving in mathematics. She had set up five centres with materials and activity cards that posed questions. One of these read, "Here is a bag of macaroni. Can you find out how many pieces the bag contains without counting the pieces individually?" The children were stymied for some time and their frustration began to mount. The teacher, busy responding to requests from other groups, did not immediately notice the difficulty they were encountering. Passing near the table later, she reacted by picking up the nearest container, a half-size dixie cup, and placing it in front of the children with the question, "Will this help?"

The group immediately began to count the number of pieces needed to fill the dixie cup. They then proceeded to find out how many times they could fill the dixie cup from the bag of macaroni. One girl recorded the results using a tally system 𝍸 𝍸 ||, *etc. The final result was 40 cupfuls. One cup held 57 pieces. No one raised the question of whether the dixie cup was an appropriate unit.*

Having found the number of times the unit was repeated and the number of pieces in the unit, the group were at a loss as to what to do next. Finally one ten-year-old requested permission to use the chalkboard. She proceeded to write until she had thirteen 57's in one column. She was half-way down a second column when the bell rang for recess.

57
57
57
.
.
.

While the children were outside, the teacher wrote ten multiplication questions on the chalkboard, including $\begin{smallmatrix}57\\ \times 40\end{smallmatrix}$. *After recess, she asked the boys and girls to do the questions. Every child in the class got the correct answer to* $\begin{smallmatrix}57\\ \times 40\end{smallmatrix}$ *yet no one from the group doing the investigation with the macaroni realized that they had found the answer to their problem. For them, finding the answer to 40 x 57 was apparently a ritual that had nothing to do with the real situation of finding out how many pieces of macaroni were contained in the bag.*



Every child needs to be proficient in each of the four operations; this requires practice. Decisions about the amount, frequency, and type of practice should be made jointly by the teacher and child. The games below provide sample activities that children have found enjoyable for maintaining and sharpening their computational skills. These games should be supplemented with variations and extensions that will accommodate the wide range of interests found in the classroom.

Link Games

The following games – *number snap, add-one, make a number*, and *happy families* – are all examples of link games. In these games players establish links between patterns of dots and the corresponding numerals. They are played with *link cards,* as described below.

Link Cards

Link cards comprise the following:

- a set of 40 numeral cards (each numeral from 1 to 10 is marked on 4 cards)
- a set of 40 pattern cards (each pattern is marked on 4 cards)

and so on to 

- a set of 40 unit cards marked with the numeral 1

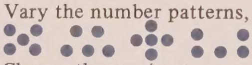
Number Snap

Objective:
To associate a numeral and an equivalent number pattern.
Materials:
Link cards (numerals and patterns to six)
Number of Players:
Two to six, and a dealer
Rules:

- Deal the numeral cards to the players (4 to 10 cards).
- Place the pattern cards face down in the centre of the table.
- The dealer turns over a pattern card.
- The first player to find the corresponding numeral in his hand says "snap". He matches the two cards and scores 1.
- If a player makes a mistake, he loses 1; the player who identifies the mistake scores 1.
- The process is repeated until a player goes out; he then becomes the dealer and a new round begins. The game continues for a specified time or number of rounds.
- The player with the highest score wins.

Variations:

- Extend the numerals to ten, or beyond.
- Vary the number patterns, for example:



- Change the scoring to correspond to the numeral in each "snap".
- Mix the pattern and numeral cards.
- Have the players make up new rules.

Add-One

Objective:
Given a whole number, to recognize the next larger whole number.
Materials:
Link cards (numeral and unit cards)
Number of Players:
Three to six, and a dealer
Rules:

- Deal each player a hand (4 to 10 cards).
- Place the stack of unit cards in the centre, face up.
- In response to the dealer's command , "Cards!", each player puts a card face up in front of him.
- The first player to notice the next larger number to his own says,"Add one" for example, if his card is 4 and he sees a 5, he says "add-one", takes the 5 and a unit card, and builds the addition bond $\boxed{4}$ $\boxed{1}$ $\boxed{5}$ in front of him. All the other players discard their unused cards.
- The first round continues until a player goes out; he then becomes the dealer and the game continues with a new round.
- Each bond scores one.

Variations:

- Use pattern cards, numeral cards, or both.
- Score by the sum of the bonds.
- Have the players make new rules.

Make A Number

Objective:
To identify two addends of a whole number. To identify two numbers whose difference is a given number.
Materials:
Link cards (numeral cards)
Number of Players:
Three to six, and a dealer
Rules:

- Deal each player a hand (4 to 10 cards) and place the rest of the deck in the centre, face down.
- On the command ,"Card!", each player places a card in front of him, face up; then the dealer turns over a card.
- The first player to build an addition or subtraction bond equal to the dealer's card, by using a second card from his own hand or from one of the other cards on the table, says ,"make a number". Example: the dealer turns up a $\boxed{5}$, player C uses his own $\boxed{3}$ with a $\boxed{2}$ from player A to build a $\boxed{3}$ $\boxed{2}$ equal to $\boxed{5}$
- Each bond scores 1.
- The first round continues until a player goes out; he then becomes the dealer and the game continues with a new round.

Variations:

- Use only pattern cards, or both numeral and pattern cards.
- Permit addition only and use any three exposed cards to make a bond.
- Permit subtraction only and use any three exposed cards to make a bond.
- Award a bonus to the player who goes out in each round.
- Have the players make new rules.

Happy Families

Objective:
To recognize the missing addend of a sum.
Materials:
Link cards (numerals, possibly extended to 20)
Number of Players:
Three to six
Rules:

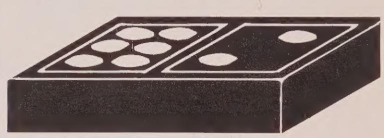
- Player A deals each player (including himself) a hand (4 to 10 cards).
- Player A places a card from his hand (say, 3) on the table and decides on another larger number (say,11). He says, "I want a number to go with my 3 to make 11." Player D answers first, saying, "Here is an 8 to go with your 3 to make 11." Player D scores 1.
- Player D continues, placing a card (say, 5) on the table. He says, "I want a number to go with 5 to make 9."
- The game continues until one player goes out.

Variations:

- Use subtraction instead of addition.
- Use only pattern cards, or both numeral and pattern cards.

Dominoes

This game can be modified in many ways to provide relevant experiences for children at different stages of growth. Small dominoes of commercial sets are often difficult to handle for young children; for this reason, as well as to introduce numerals, it is useful to make your own dominoes, using 1" x 2" pine or spruce cut into 4" lengths and sanded smooth. When labelled and varnished, they are more attractive and durable.



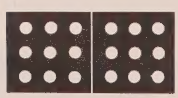
Use dots to label one set and numerals for the other. A pattern for enlarging the sequence of dominoes is given on page 5.

The number of dominoes used and the rules of the game can be adjusted for specific purposes. The following are examples of a few of the many possibilities.

Game 1

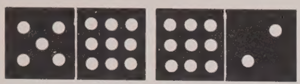
Objective:
To recognize equivalent number patterns from zero to thirteen.
Materials:
Pattern dominoes
Number of Players:
Two to four
Rules:

- Place all the dominoes face down on the table. Each player chooses the same number of dominoes. These are placed on edge in front of each player so that the labels are visible only to him.
- The player with the largest pair, such as

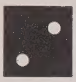


places his domino face up in the centre of the table.

• The player on his left then places a domino with a matching number of dots on either end of the first domino, as shown:



• The next player plays a domino that will match either the

 or 

If he does not have one in his possession that matches either of these, he draws a domino from the 'bank'.

• If the domino drawn is not playable, he loses his turn.

• The player who succeeds in getting rid of all his dominoes first is the winner.

Game 2

Objective:
To recognize equivalent numerals and number patterns.

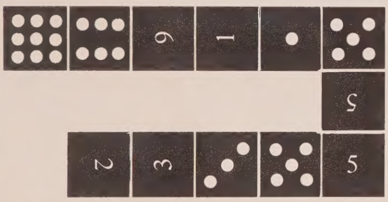
Materials:
Pattern dominoes and numeral dominoes

Number of Players:
Two to four

Rules:

• Similar to Game 1, except that player 1 chooses a domino with numerals, player 2 selects a domino with dot patterns, player 3 chooses numerals, and so on.

• The challenge is to match the number of dots on one domino with the correct numeral and vice versa.



• The winner is the player who succeeds in playing all his tiles first.

Game 3

Objective:
To recognize equivalent numerals and number patterns; to recognize addends of a given number.

Materials:
Pattern dominoes and/or numeral dominoes

Number of Players:
Two to four

Rules:

• The procedure is the same as for Game 1, except that tiles may only be added if the sum of the symbols on the joining halves is ten (or some other number chosen by the children or the teacher).

• This time the zero or blank domino can be treated as 'wild' or 'free' – i.e., it assumes any value the child chooses.

• Again, the first player to use all his tiles is the winner.

Variations:

• The numerals used as labels can represent larger numbers such as 12, 13, 14, etc.

• Further challenges can be imposed by restricting sums to either odd or even numbers.

Game 4

Objective:
To identify number pairs with a fixed difference.

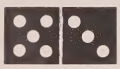
Materials:
Pattern and/or numeral dominoes

Number of Players:
Two to four

Rules:

• The procedure is the same as in Game 1.

• The criterion for playing a domino is that the difference between the domino being played and the one on the table must be 3 (or some other preselected number). For example: if the domino is



the player must choose

 or 

to play on 

or

 or 

to play on

• Again, the object of the game is for each player to make plays that cause the opponents the most difficulty while disposing of his/her own tiles as quickly as possible.

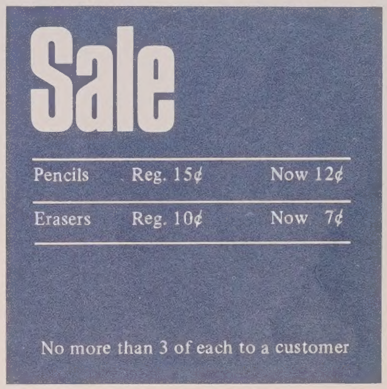
Behind the Story

A. Once in a while it is useful to make up situations or stories to fit computational exercises.

Take number sentences such as:
$5 + 4 = 9 \quad 4 + 5 = 9 \quad 9 - 5 = 4$
Make up a story to show how these number sentences might have originated. For instance: I read five pages in my book this morning and four more this afternoon; I have read nine pages today. Jean read four this morning and five this afternoon; she also read nine pages. George has read five pages up to now; he must read four more to catch up.

B. Simple problems can easily be made more challenging by altering some of the conditions.

Mary wanted to buy a pencil and an eraser. She knew that the pencil cost 15¢ and the eraser 10¢. She had three dimes in her purse. When she arrived at the store she saw this sign.



• How much will Mary save?

• Could she buy two pencils and one eraser with her 30¢?
What about one pencil and two erasers?

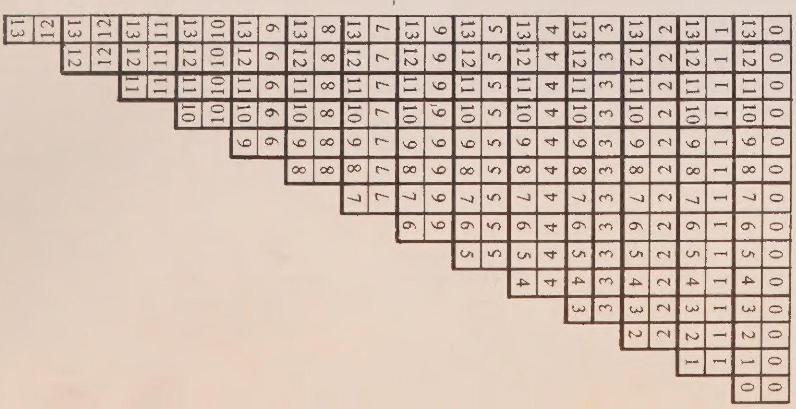
• If she had one dollar to spend, what is the largest purchase she could make? How much change would she get? What is the smallest possible purchase?

• How many different purchases could be made at this store with the limitation that has been put on this sale? We already know the largest and the smallest. How many are in between? How much would each of these purchases cost?

• The largest would be 57¢. What would the next largest be? Could there be a purchase of 54¢?

• Mary's friend, Dorothy, spent 24¢. What did she buy? Can you be sure that it was two pencils?

• Peter heard about the sale. When he



gave the cashier 50¢, he received 5¢ in change. What did Peter buy?
• The cashier had made a 'work saver' chart that looked like this:

Number of pencils @ 12¢

Number of erasers @ 7¢		0	1	2	3
	0				
	1				
	2				
	3				

Complete the chart. (Look for short cuts.)
• Make up a chart using the regular prices for the pencils and erasers.
C. Similar problems can be developed using advertisements from newspapers and sale catalogues. These can be varied in difficulty by using fractional or per-cent reductions and changing the restrictions on the quantities allowed each customer.

Shopping Spree
Have the children cut out pictures of groceries, clothing, toys, and other products from an old catalogue. Make a "store" booklet by pasting these on the pages and showing the prices of the articles. (Simplify, if necessary.)

(i) *Problem*
Ask each child to make up a shopping list and exchange it with a partner. Each child "shops" and calculates his expenditure.

(ii) *Problem*
Prepare a supply of 'playmoney'. Each child draws a ticket that tells him how much money he has to spend. He obtains the playmoney from the bank, simulates the shopping, pays for his purchases, and calculates the change due to him.

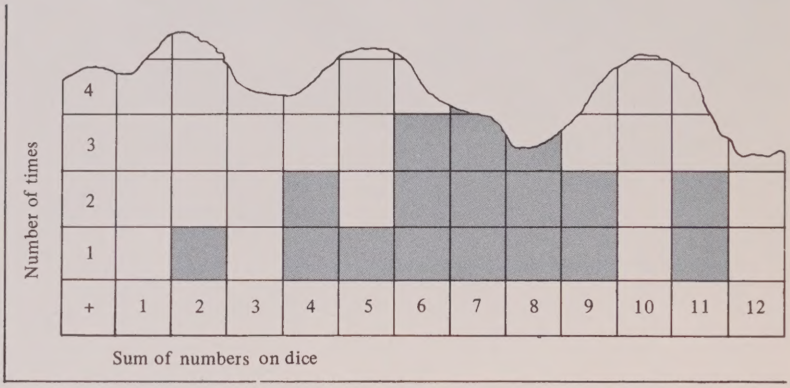
(iii) *Problem*
Each child draws a ticket stating how much money she has to spend. She is to decide how many and what variety of articles she can afford to buy with this sum.

The computation in these problems can be varied in difficulty by:
(a) varying the number of articles to be bought, from two upward (e.g., 2 cubes at 10¢);
(b) varying the complexity of the prices, from 10¢ or 15¢ or 25¢ for each item (e.g., 10¢ and 15¢ [mixed prices]; 10¢ and 15¢ and 25¢ [mixed prices]; $1.50, $1.25, $1.15 [mixed prices]; $1.48, $1.22, $1.16 [mixed prices]).

Roll the Dice
Materials
Squared paper and dice – The dice may be purchased commercially or made from wood or plastic cubes of suitable size. Either dots or numerals can be used as labels.

Method
Working in pairs or in small groups, the children decide on an operation to be applied to the pair of numbers. For example: 3 + 2 or 3 - 2 or 3 x 2.
Roll the dice 100 times and record the scores on the squared paper.

Taking turns, roll the dice 100 times and record the results by means of a graph. This activity not only provides practice in recalling basic facts but gives experience with the order of operations. Children should be encouraged to predict the most frequent and least frequent results and to give reasons for their predictions.



• Which sum or sums occur more frequently?
• Which occur least often?
• Compare your results with those of your classmates.

The teacher can supply the squared paper with the baseline and the columns already numbered if the children need this kind of help. Later on they can find out for themselves how many columns are needed.

Variations
1. Change the numerals on the faces of each dice. For example, the faces on one dice could be labelled 2, 4, 6, 8, 10, 12, and the faces on the other 1, 3, 5, 7, 9, 11. Taking turns, roll the dice 100 times and record the results on a graph. Compare your results with those in the first game. Which sum or sums occur most often? Which occur least often? What do you notice about the sums?
There are many possibilities for labelling, such as multiples of 3, 5, or 10, square numbers, prime numbers, and others.

2. Use two dice, using 0 in the place of the 6. Ask the children to predict what will happen when:
• you add 0
• you multiply using 0
Repeat the 100 rolls, record the results on squared paper, compare this graph with your first graph. Which sum occurs most frequently? Which occurs least often?

3. Use three different dice with various combinations of numerals. The operation may be addition, subtraction, multiplication, or a combination of these. For example: using red, blue, and green cubes, add the numbers appearing on the red and blue dice and subtract the number on the green.

From Fragments to Patterns
The examples below illustrate conventional forms of writing addition exercises:

4	3	4	4	5	3	6	3	5	5
+3	+6	+5	+6	+3	+4	+4	+5	+6	+4
7	9	9	10						

4 + 3 = 7 6 + 3 = 9 5 + 4 = 9, etc.

Although children should be able to do such exercises efficiently, these forms do not help them to see patterns that substantially reduce the number of facts to be memorized, nor do they reinforce understanding of number properties.
The following sequence of exercises represents one method of improving the child's addition skills while stimulating his curiosity and increasing his enjoyment.

1. Add the given numbers. Place the sum in the empty square.

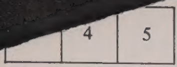
4	5	

(a)

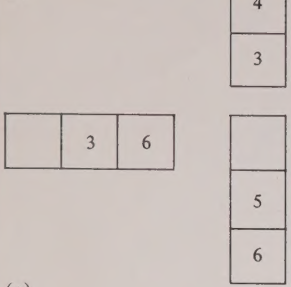
4
3

3	6	

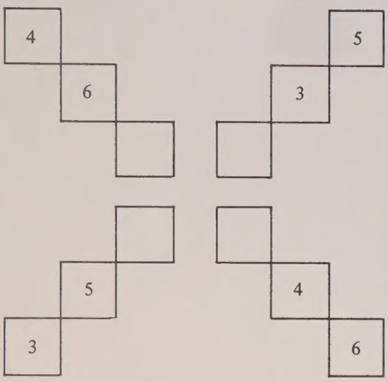
5
6



(b)



(c)



Notice that these examples cover the same 12 number facts as the conventional exercises above.

2. In the example below the same number facts are compressed into a 4 x 4 box.

Fill in the empty squares by adding horizontally, vertically, and diagonally.

	4	5	
	3	6	

The completed form takes up little space, uses the 'puzzle' format to stimulate interest, and, when completed, offers the satisfaction of accomplishment. It is also a compelling demonstration that order does not make any difference in determining the sum of two numbers.

There is much to learn from this form. The discussion could begin with "Do you see anything in this structure worth talking about? " Responses will include such statements as, "The opposite corners are alike.... The middle two pairs at the top and the bottom are the same.... The middle pairs on the vertical sides are alike." Perhaps someone will notice that the sums are from the sequence 7, 8, 9, 10, 11.

The question "Is this a special case, or will patterns like this one occur again? " leads to further experiments with different arrangements of the same numbers and with different numbers in the central four squares, as illustrated below.

	3	4	
	5	6	

	6	7	
	8	9	

There are many obvious similarities. A new pattern was created when the arrangement was changed, namely, the numbers in all four corners are the same. Using different numbers, the sum of the 'middle pairs' is 30.

The class is now ready for individual activity. The children will want to try their own versions. Some may want to try bigger numbers, others may wonder what happens if zero is included in one or more of the central squares. Do the numbers have to be in sequence to make the numbers in all four corners the same? Can you find new patterns? Can you find examples that are without pattern?

There are some ingeniously organized arrays of numbers which have interested mathematicians and fascinated children of all ages for thousands of years. They are called 'magic squares' because the sums of every row, every column, and both diagonals are equal. The following are examples of *3 x 3* and *4 x 4* magic squares.

(a)

6	1	8
7	5	3
2	9	4

(b)

13	2	3	16
8	11	10	5
12	7	6	9
1	14	15	4

(c)

1	10	9	14
13	10	5	6
8	3	16	7
12	11	4	7

A standard magic square uses consecutive natural numbers starting with 1. Example (a) is a standard magic square of the third "order", (b) is standard and of the fourth "order", (c) is a non-standard magic square of the fourth "order".

Once a magic square has been constructed, more magic squares can be made from it by adding, subtracting, multiplying, or dividing every number in the square by the same number.

6	1	8
7	5	3
2	9	4

+5

11	6	13
12	10	8
7	14	9

÷4

1½	¼	2
1¾	1¼	¾
½	2¼	1

−1

5	0	7
6	4	2
1	8	3

x 2

12	2	16
14	10	6
4	18	8

Other variations can be made by reflecting the square either on the vertical or horizontal axis or by rotating it.

6	1	8
7	5	3
2	9	4

Rotate half turn about the centre of the square

4	9	2
3	5	7
8	1	6

Reflect in diagonal

4	3	8
9	5	1
2	7	6

Reflect in vertical axis

8	1	6
3	5	7
4	9	2

Rotate quarter turn clockwise about the centre of the square

2	7	6
9	5	1
4	3	8

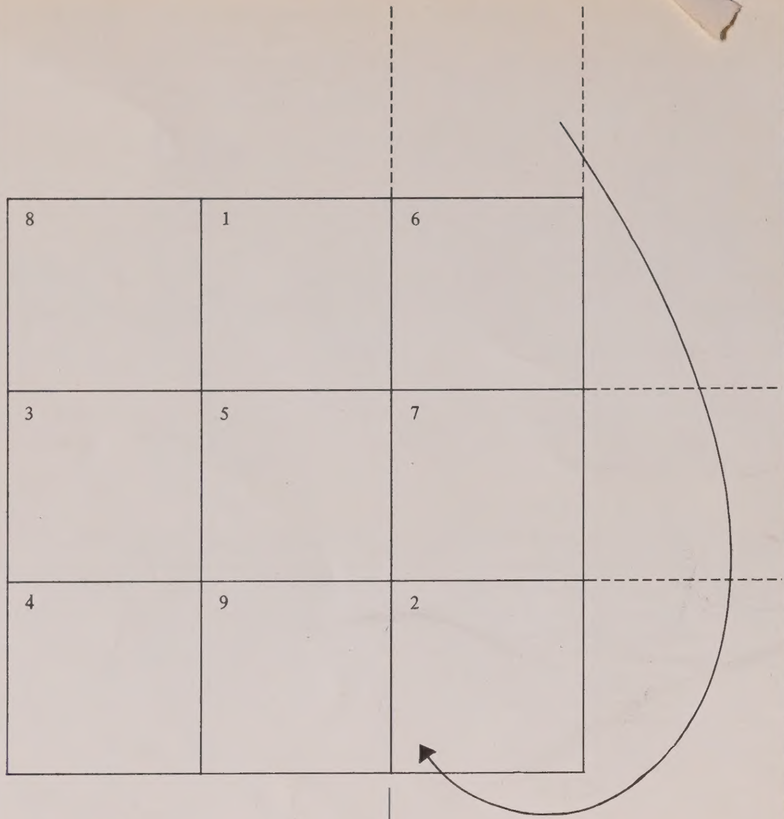
Find other variations.

At the beginning, the children will be interested in testing the squares to see if they are indeed magic. Later they will be able to fill in missing numbers in magic squares, provided that at least four numbers are given, three of them in a straight line. Another way is to locate three numbers, two of which must be in a straight line, and provide the sum of one row, column, or diagonal.

4		
3	5	7

11		
12		
		9

Sum = 30



Finally, some childr ... imes their own magic square... and error or by following t...

1. Write the numeral 1 in the mid... unit of the top row.

2. Move right one and up one. This position is above the third column; enter the 2 in the bottom square of the third column.

3. Move right one and up one. This position is to the right of the second row; enter the numeral 3 in the left square of the second row.

4. Move right one and up one. This square is filled; enter the 4 in the square below 3.

5. Move right one and up one; enter the numeral 5 in this square.

6. Continue the pattern of steps 2 to 5 until all the squares have been labelled.

7. Use this order to enter any sequence of numbers that you choose. For example: 3, 6, 9, 12, 15, 18, 21, 24, 27.

This pattern can be extended to squares of order 5 and 7. It will not work for magic squares of order 4 or 6.

Identification no. 74-75/4062